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TOPICAL REVIEW

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Topical Review

Using flexural MEMS to study and exploit nonlinearities: a review

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Abstract

This review describes recent experimental developments in the study of nonlinearities for flexural microelectromechanical systems (MEMS) structures. It includes motivation for studying a variety of nonlinear phenomena, highlighting applications where nonlinearities are relevant. Examples are described where nonlinearities are seen as a nuisance, along with techniques used to mitigate the nonlinearities. More recently, efforts to exploit nonlinearities for improved performance have emerged and are also described. After a summary of the theory behind nonlinear mechanisms, the article describes progress in several areas of nonlinearity: engineering nonlinearities to enhance performance, bifurcations and multistability, parametric processes, frequency mixing, nonlinearly coupled resonances, and nonlinear dissipation.

Kewyords: nonliniearity, flexural, Duffing, parametric resonance, coupled resonators, phase noise, bifurcation

(Some figures may appear in colour only in the online journal)

1. Introduction

1.1. Why study nonlinearities?

Generally speaking, one decides to study a topic because it either possesses useful properties that can be exploited, or it creates negative effects that must be mitigated. In the case of nonlinearities in microsystems, there is a bit of both. Studying nonlinear behavior is particularly important for micro- and nanoscale devices, as it emerges rapidly when devices are miniaturized to these size scales [1, 2]. Research on nonlinear dynamics in micro- and nanosystems has been a growing area for nearly two decades. For a summary of research in the first decade, readers are referred to three thorough reviews from that time period [3–5]. During the last decade, understanding of nonlinear phenomena at the micro/nanoscale has continued to advance. Additionally, techniques that use nonlinearity advantageously have been explored, and new application spaces have begun to emerge. The following review details reasons why one would (or would not) want nonlinearity in microelectromechanical systems, provides background on nonlinearity in microelectromechanical systems (MEMS), and summarizes prior research organized by the type of nonlinear process. For brevity, this review focuses on recent experimental work pertaining to nonlinearity in flexural structures, which are widely studied and utilized in MEMS devices.

1.2. Nonlinearities can be harmful

1.2.1. Frequency stability. One of the most obvious problems of nonlinearities in flexural MEMS is the resulting dependence of resonant frequency with amplitude, known as the amplitude-frequency effect (A-f effect). Many of the applications of these devices depend on operating in a resonant mode at a particular frequency, including timing devices [6], frequency filters [7], resonant accelerometers [8], resonant energy harvesters [9], and gravimetric sensors [10]. For one example, nonlinearities can limit the power handling of radio frequency (RF) front end filters by causing shifts in their resonant frequency and motional impedance, leading to distortion of the passband [7].

1.2.2. Phase noise in oscillators. A high signal-to-noise ratio is desirable for reduced phase noise in oscillators. As the drive level is increased, nonlinear effects will at some point become significant, which is a manifestation of the aforementioned A-f effect. This dependence eventually leads to a bifurcation in the amplitude-frequency curve, which typically limits the amplitude at which the oscillators are operated since the frequency of a device is dependent on its history beyond this limit [11]. This limitation on amplitude places a maximum on the drive current that can be applied, which results in a relative increase in far-from-carrier phase noise [12]. Also, the dependence of frequency on amplitude converts amplitude noise to near-carrier phase noise [13, 14]. Finally, nonlinear coupling can allow out-of-band interferers to couple into the band [15].

1.2.3. Sensors. Many common MEMS sensors (e.g. accelerometers, gyroscopes) are operated in the linear regime in order to avoid hysteresis and additional noise associated with nonlinearities. From a purely practical sense, nonlinearities can make interpretation of sensor output more complicated as in the case of electrostatic detection of large displacements, a common occurrence for MEMS gyroscopes [7]. Also, emerging systems that couple mechanics with other phenomena, such as optics [16], bring along new classes of nonlinearities to be considered. Furthermore, at small size scales it is important to understand nonlinearities because they can often establish the fundamental limits of detection for a variety of sensing platforms [17].

1.3. Nonlinearities can be helpful

1.3.1. Phase noise in oscillators. While the previous section asserts that nonlinearities lead to increased phase noise in oscillators, there have been observations that operating at specific points in the nonlinear regime *decreases* oscillator phase noise [18, 19]. Alternatively, nonlinear coupling of resonant modes in a single structure [20], or coupling of separate resonant structures [21], can stabilize the amplitude, and therefore frequency, of a desired mode.

1.3.2. Frequency stability. Again, there are both harmful and helpful aspects to nonlinearities. The transition from quartz resonators to silicon resonators has driven efforts for new ideas in minimizing the temperature coefficient of frequency because, unlike quartz, silicon does not have a crystalline orientation with zero temperature coefficient of frequency. A range of techniques for enhancing frequency stability are described below [22, 23].

1.3.3. Instruments for scientific discovery. In many cases, micromechanical devices can be used as instruments to investigate small-scale phenomena, such as the Casimir force [24] or fundamental limits dictated by coupling of Brownian motion between resonant modes [25]. In many cases, widely used linear models begin to break down at small scales and need to be reformulated [26]. Additionally, atomic force microscopy uses the nonlinear van der Waals force to map surface morphology, magnetism, charge, and other properties with extreme precision [4].

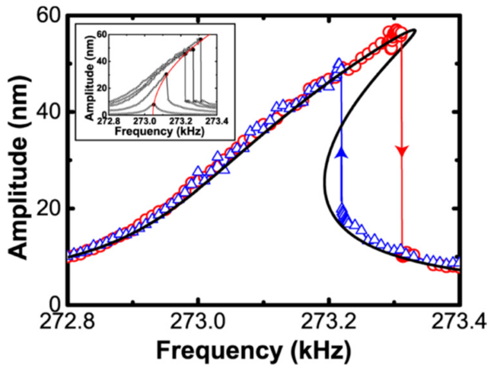

Figure 1. Duffing nonlinearity beyond the critical amplitude. Notice the hysteresis between increasing (red circles) and decreasing (blue triangles) frequency sweeps. (Inset) Resonant frequency increasing at higher drive powers. The red curve plotted through the peaks of the curves is known as the 'backbone curve'. Reprinted with permission from [31]. Copyright 2013 American Chemical Society.

1.3.4. Sensors and other devices. The pull-in phenomenon of electrostatically driven MEMS is desired in some devices, such as MEMS switches, and can be analyzed to reduce switching voltage [27]. Additionally, the fundamental quadratic relationship between voltage and electrostatic force is frequently used to shift noise out of band, thereby improving resolution of displacement measurements [28]. Nonlinearity can also be used for mechanical computation [29]. In gyroscopes, improvements can be observed in some parameters, such as angle random walk and bias instability, even when operated above the critical bifurcation amplitude [30]. Finally, extreme sensitivity has been observed in parametric sensors [3].

2. Nonlinear forces in MEMS

2.1. An illustrative example: the Duffing oscillator

The dynamics of a forced oscillator depend on the sum of three forces: (1) a restoring force represented by Hooke's law, $F_{res} = -kx$, (2) a driving force that is proportional to some external stimulus, $F_{drv} = Af_d \cos(\omega t)$ where f_d is the applied stimulus amplitude (such as an applied electric field), and (3) a dissipative force, $F_{dis} = b\dot{x}$. The forces can have a variety of origins (mechanical, electrical, magnetic, etc). Starting from Newton's Law, the equation of motion for a forced oscillator is

$$m\ddot{x} + b\dot{x} + kx = Af_d \cos(\omega t), \tag{1}$$

where $x, \dot{x}$, and $\ddot{x}$ are the displacement, velocity, and acceleration of the system, respectively. The distinguishing factor of a nonlinear system is that at least one of the force coefficients (m, b, k, or A) are not constant, but are instead functions of the dynamic variables of the system, such as the velocity, position, stimulus amplitude, or elapsed time.

The most common nonlinear MEMS device is that of the Duffing oscillator [1, 4, 9, 11, 14, 17], where the oscillator spring constant depends quadratically on its displacement. This leads to the nonlinear equation of motion

$$m\ddot{x} + b\dot{x} + kx + \beta x^3 = F_{drv}, \tag{2}$$

where β is the cubic nonlinearity term that results from the nonlinear spring. Because of the nonlinearity, the resonant frequency now varies quadratically with the oscillator amplitude (figure 1 inset). For negative values of β the resonant frequency will continually decrease with increasing force, known as spring softening. For positive values the resonant frequency will increase, leading to spring hardening. Beyond a critical amplitude ($x_c = \sqrt{\frac{32}{9\sqrt{3}}\frac{1}{|\beta|Q}}$, where Q is the quality factor) the frequency response develops a bifurcation and becomes multivalued, as shown in figure 1.

The most distinguishing feature of an experimental measurement of the Duffing nonlinearity is that the concave portion of the theoretical frequency response is inaccessible. This branch is unstable and approaching it from one stable branch causes the resonator to jump to the other stable branch. As a result, the frequency response now has hysteresis because adiabatically sweeping frequency up is different from when sweeping frequency down, which is not possible in a linear system. However, this is only true for an open-loop measurement. Using closed-loop control, the feedback stabilizes the oscillator and makes the full frequency response curve accessible [32, 33].

2.2. Sources of nonlinearity

Nonlinearity in MEMS systems can come from a large variety of sources. Mechanical nonlinearities have two origins, intrinsic material effects and geometric nonlinearity [11, 34], and can be modelled with a displacement dependent spring constant.

$$k = k_0\left(1 + k_1 x + k_2 x^2 + \ldots\right). \tag{3}$$

For flexural devices k_1 is generally zero due to the symmetry of the structures, but can be nonzero for other geometries, such as bulk acoustic wave resonators [11]. Geometric nonlinearity dominates when there is large displacement, as the structure deformation can no longer be ignored and leads to a dynamic change in the spring constant that is a function of the displacement. Due to their large displacements, geometric nonlinearity is one of the most significant sources of nonlinearity in flexural MEMS devices [31, 34]. The nonlinear coefficients can be predicted by classical Euler–Bernoulli beam theory [11, 35], but the assumptions do not always hold for all mode shapes [26]. Geometric nonlinearity can be tuned by using a composite structure, as was done by Asadi *et al* [36]. In this work, the authors studied a silicon cantilever that is anchored to the substrate with a polymer structure. The elastic anchor undergoes significant tension during resonance and introduces the nonlinearity in the system. Structural design can also be used to do the reverse and remove nonlinearity, as was done by Chen *et al* [37]. In their design of clamped-free semicircular beam resonators, the free end is engineered to release its axial tension and eliminate the cubic nonlinearity. Material effects manifest as higher order stiffness constants and are more prominent in bulk mode devices [11], as their displacements are too small for geometric nonlinearity to significantly contribute on its own. In silicon, these material nonlinearities have been shown to be strongly dependent on doping and crystal orientation [34].

Another major source of nonlinearity in MEMS resonators is the electrostatic force. The electrostatic force between two parallel plates is given by [11, 12, 38]

$$F_{es} = \frac{\epsilon A V^2}{2(d-x)^2} \approx \frac{\epsilon A V^2}{2d^2}\left(1 + \frac{2}{d}x + \frac{3}{d^2}x^2 + \frac{4}{d^3}x^3 + \ldots\right), \tag{4}$$

where ϵ is permittivity, A is the plate area, and d is the initial spacing between plates. This force is proportional to the square of the voltage, making it intrinsically nonlinear. Superimposing a large DC bias on top of the AC drive signal is a commonly used technique that approximates linear behavior, similar to the way that transistors can be biased to a quiescent point. However, this nonlinearity can be desirable as a method to cancel parasitic feedthrough signals during device operation [28, 39]. The DC bias for resonant beams is typically applied in a way to balance the force between symmetric electrodes, thereby removing the even terms in equation (4). As a result, the cubic term is often the major contributing factor for the electrostatic nonlinearity [14, 38], though higher order terms can come into play for high bias voltages or high drive amplitudes [23, 40]. This cancellation of the symmetric term also reduces the effect of the pull-in instability [41], where the DC bias voltage overcomes the mechanical restoring forces and causes the two electrode plates to come into contact. Electrostatic forces also lead to a dynamic pull-in instability when large AC voltages are involved, typically requiring a much lower voltage amplitude than the static case [42]. In silicon resonators the dominant nonlinearity can be readily determined, as the electrostatic nonlinearity leads to spring softening and the geometric nonlinearity leads to spring hardening [12].

A large number of other factors can contribute to nonlinearity, depending on the device structure and environment. Devices with small actuation gaps experience nonlinear surface forces (such as the Casimir force) that begin to significantly influence the dynamics [24, 43]. For example, recent work on resonant switches has shown the ability to tune the nonlinear, repulsive van der Waals contact force by modifying the surface coating [43]. Variation in temperature has been found to change the strength of nonlinearity, through the temperature dependence of a device's material properties [8]. Optical interactions in the characterization of opto-mechanical devices can lead to a number of nonlinear processes [16]. Like electrostatic forces, magneto-mechanical forces are also intrinsically nonlinear [44].

2.3. Effects of scaling on nonlinearity

As discussed above, mechanical nonlinearity in flexural devices is often geometry dependent [12]. Following Kaajakari's treatment of a clamped-clamped beam resonator [11], the critical value of input energy for a high Q resonator is

$$E_c = \int_0^{x_c} k(x)\ xdx = \frac{1}{2}k_0 x_c^2\left(1+\frac{1}{2}k_2 x_c^2\right) \approx \frac{1}{2}k_0 x_c^2 = \frac{1}{254.9}\frac{\left(\rho\omega_0^2\right)^{\frac{5}{2}}}{QY^{\frac{3}{2}}}hL_0^7 \tag{5}$$

where ω_0 is the linear resonant frequency, ρ is the material density, Y is the Young's modulus, h is the beam height, and L_0 is the beam length. It is not difficult to see that, for the same frequencies and materials, resonators become dramatically easier to drive into nonlinearity as they scale down in size. This limits the dynamic range of the resonator and becomes an issue for nanomechanical sensors [1]. For very high aspect ratio devices, such as nanotube or nanowire systems [17], this is especially detrimental as thermomechanical noise increases with miniaturization and limits the dynamic range on the low end.

As dimensions shrink to the micron-scale and below, size effects due to material inhomogeneities, such as dislocations and polycrystallinity, cannot be ignored and begin to influence the mechanics [45]. Measurements by Lam *et al* of the deflection of epoxy micro-cantilevers found that the normalized bending rigidity increases for thinner beams due to these size effects [46]. Similar experiments were done by McFarland *et al* on injection molded polypropylene MEMS cantilevers, again finding that micron-scale beams were stiffer than expected [47]. Liu *et al* studied microscale copper wires of various diameters under torsion, revealing that the normalized initial yield torque increases for smaller diameters [48]. Tang *et al* applied micro/nanoindentation measurement and atomic force microscopy to investigate length-scale factors for silicon cantilevers [49, 50]. Models incorporating size effects have found that they have a significant effect on resonator dynamics. Analysis by Dai *et al* on nonlinear cantilevers demonstrated that increasing the length-scale parameter, which represents the size of the material microstructure, causes the stiffness, and therefore resonant frequency, to increase [51]. Work by Farokhi *et al* showed that resonant beams with an initial static deflection can actually switch from the softening type of nonlinearity to the hardening type if the size effect is strong enough [52]. In electrostatic devices, the increased stiffness from the size effect results in the effective strength of the electrostatic nonlinearity to be weaker. Models incorporating the size effect predict higher voltages for the electrostatic pull-in instability, matching experimental data much better than classical mechanical models [53, 54].

3. Analytical approaches

While this review is focused on experimental work on nonlinearity in MEMS, it is worthwhile to briefly inform the reader about analytical approaches that have been used to address problems in nonlinear MEMS. By far the most popular approach is to start with Euler-Bernoulli beam theory and to then reduce the problem to something tractable using the Galerkin method for discretization [55–57]. A variation of this was used by Kacem *et al* to study bifurcations due to Duffing nonlinearities, using the method of averaging and a formulation of the electrostatic force that includes fringing fields [56, 58]. Besides Euler–Bernoulli theory, several others elected to use variational methods to derive the equations of motion of the MEMS structures [17, 40, 59]. In the case of a beam with appreciable shear deformation, one author elected to use Timoshenko beam theory, while also incorporating thermal effects to investigate thermal elastic dissipation (TED) [60].

Another popular approach that has been used instead of, and in conjunction with, the previously discussed methods is the method of multiple time scales [23, 40, 61, 62]. This approach separates the problem into two times variables, one fast moving variable that captures the dynamics of the oscillator and a slow moving one that captures the amplitude variations. Mechanical theories incorporating length-scale parameters, such as strain gradient elasticity and modified couple stress theory, have also been used in tandem with Euler–Bernoulli beam theory to study the influence of size effects in nonlinear resonant beams [51, 52, 63–65]. Outside of purely mechanical models, an electrical model for a nonlinear MEMS oscillator was developed, incorporating not only MEMS nonlinearities but nonlinearities in the oscillator circuitry as well [66]. Atalaya *et al* presented a sophisticated model of nonlinearity in nanomechanical systems, using a quantum approach to study decoherence and dissipation of low frequency eigenmodes [67].

4. Research on nonlinear dynamics

4.1. Engineering nonlinearities to enhance performance

Initial work in engineering nonlinearities was generally focused on ways to cancel nonlinearities or otherwise reduce their impact. That remains an active area of research, but it has been joined by a more recent development, exploiting nonlinearities in ways which can be used to enhance performance. As will be seen below, the bulk of the work is centered around the performance of oscillators for timing devices, with additional work involving resonant gyroscopes.

Before getting into specific examples, it is worthwhile to ask the question why nonlinearity is a problem in resonators. Simply put, the resonant frequency of devices is altered when nonlinearities are introduced, typically through mechanical stiffening or electrostatic softening. Changes in resonant frequency are problematic for devices such as timing references, which need to output a stable frequency. Resonant sensors are also impacted because their frequency output should only be affected by the property being measured and not by their drive amplitude. In the field of oscillators and timing devices, the change in frequency caused by increasing nonlinearity with increasing amplitude is called the amplitude-frequency effect (A-f effect). Any amplitude noise will be converted to frequency noise by the A-f effect, which is particularly problematic for timing devices. Due to this coupling, the motivation for operating oscillators at higher amplitude for improved signal-to-noise ratio is diminished because increasing the amplitude also increases the noise.

4.1.1. Cancellation of nonlinear forces in resonators. Early work by Kozinsky *et al* experimentally demonstrated cancellation of the third order mechanical stiffening nonlinearity using the second order electrostatic softening

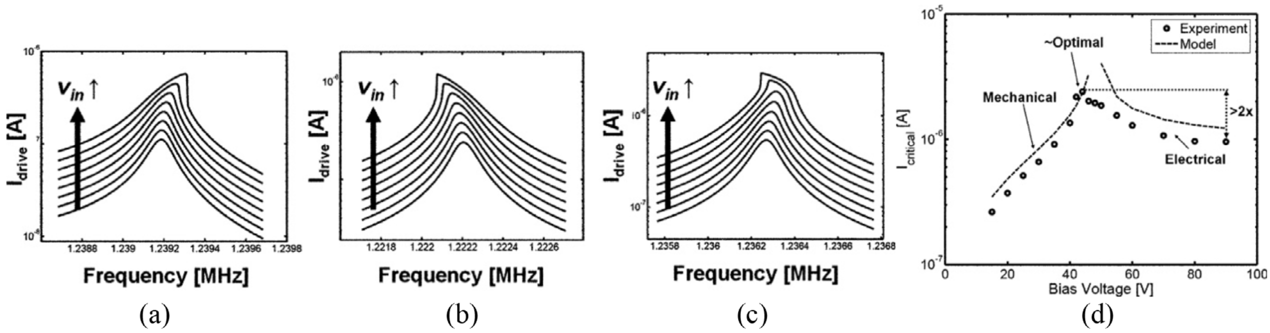

Figure 2. Tuning and cancellation of nonlinearities using a DC bias voltage. (a) mechanical nonlinearities dominate (b) electrostatic nonlinearities dominate (c) bias region where nonlinearities cancel (d) output current at the onset of Duffing bifurcation is enhanced ~2× when nonlinearities are canceled. Reprinted from [14], with the permission of AIP Publishing.

nonlinearity in nanomechanical beams, with the aim of increasing the dynamic range of nanomechanical sensors [33]. Shortly after, Agarwal *et al* explored this type of nonlinearity cancellation in order to allow for a higher drive current in MEMS resonators, as shown in figure 2 [14]. In both cases a DC bias was used to tune the electrostatic nonlinearity, which allows a sweep through regimes where mechanical and electrical nonlinearities each dominated, and find an optimal bias point in the middle. Subsequent work investigated the scaling of nonlinearities in clamped-clamped beams to provide guidelines for those wanting to design resonators [12].

Design can also be used to control the nonlinear properties of a resonant beam. In one example, shape optimization allowed for reduction of the stiffness nonlinearity, a result that was first theoretically shown [68] and then experimentally demonstrated [69].

4.1.2. Exploiting nonlinear forces to enhance resonator performance. In the past several years, interest has expanded beyond the question of how to mitigate nonlinearities, and several efforts have been undertaken to examine ways in which nonlinearity can be utilized. Much of this work has focused on improving the performance of oscillators. As described above, the term resonator will be used to describe a device operated in its open-loop mode, whereas an oscillator describes the closed-loop system comprising the resonator and its sustaining feedback circuit. When characterizing nonlinear behavior, resonators will only show part of the Duffing curve, depending on which direction the frequency is being swept. When characterizing oscillators, however, the entire curve can be mapped out by controlling the phase in the system [32]. Kenig *et al* used this fact to analyze nonlinear oscillators for regimes where the phase could be set to minimize total phase noise in the system [70]. These concepts were subsequently demonstrated in nanomechanical clamped beams by the same group [18], as seen in figure 3.

Dependence of resonant frequency on ambient temperature is well-known and very important for commercial oscillators. Even structures that are immune to thermally induced anchor strain will have a temperature coefficient of frequency (TCF) because their material properties (e.g. Young's modulus) depend on temperature. Quartz, a material traditionally used

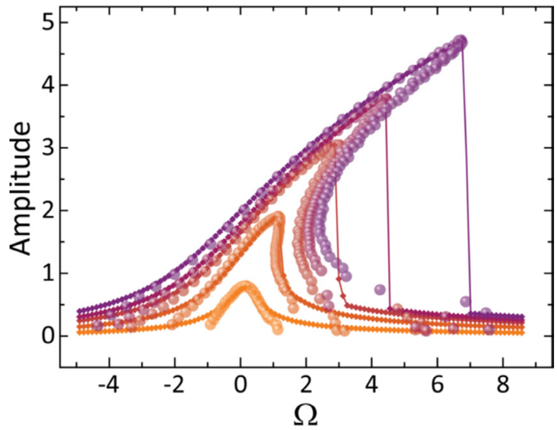

Figure 3. Resonator amplitude versus normalized frequency (Ω) at various levels of driving power. Open loop response shown by squares and solid lines, displaying Duffing at higher power levels and jump to the lower stable branch after the peak amplitude. Spheres represent results from closed loop oscillation, which can map out the entire Duffing curve, including regions with two amplitude solutions for a given frequency. Reprinted figure with permission from [18], Copyright 2013 by the American Physical Society.

for resonators, has crystalline directions that have a zero TCF and can therefore be manufactured to have low TCF for a reasonable range of temperatures. Silicon, however, has no such special crystalline direction, which has been a limiting factor for adoption of silicon resonators. One method recently demonstrated for stabilizing the resonator frequency is coupling of the nonlinear A-f effect with the temperature coefficient of quality factor (TCQ). For this work, the key concept is that a quality factor that changes with temperature translates to an amplitude that changes with temperature. Defoort *et al* showed that an operating point could be chosen such that a change in temperature would induce a change in quality factor, and the resulting change in amplitude would induce an A-f effect that would counteract the normal change in TCF. Using this method, the authors measured a 25× reduction in TCF as compared to the linear case [22].

Canceling nonlinearities in resonators, while generally useful, also gives rise to other factors that must be considered. One of these factors is the synchronization range. When a device cannot be driven at its exact resonant frequency, synchronization range is a measure of how close the driving

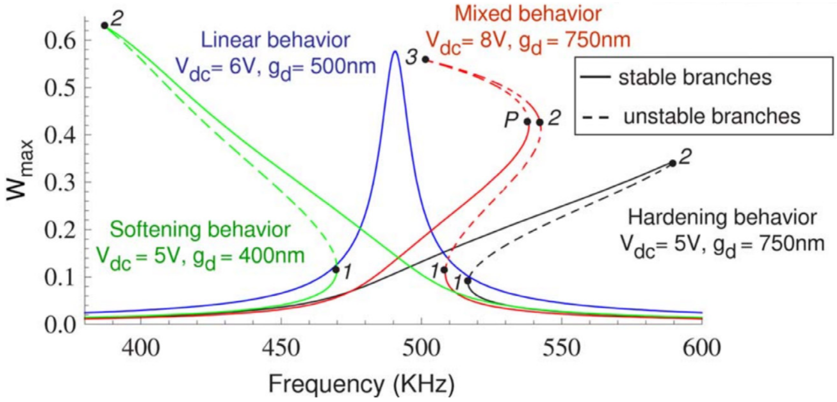

Figure 4. Different types of nonlinear behavior possible by tuning strength of spring softening and spring hardening behavior. Reprinted from [56], with the permission of AIP Publishing.

frequency must be to still excite resonance in the device. This is particularly relevant for cases where there is a desire to synchronize multiple resonators, such as improving frequency stability in timing applications. The authors of one study showed that the synchronization range grows monotonically for a single type of nonlinearity (e.g. mechanical) but decreases to a minimum when nonlinearities cancel and then grows more rapidly as higher order nonlinearities subsequently begin to dominate [23].

Canceling mechanical nonlinearities with electrostatic nonlinearities in resonators was discussed above for timing applications. This technique has also been applied to resonant quad-mass gyroscopes by Taheri-Tehrani *et al* [30]. The work investigated optimal biasing to cancel nonlinearities and reduce the amplitude-frequency effect. Surprisingly, the authors found that the angle random walk was lower where the amplitude-frequency effect was larger (i.e. in a highly nonlinear regime). Angle random walk is a property related to the noise that is viewed as a buildup of error in the sensor readout of angle over time. The fact that the noise actually increased at the point where electrostatic and mechanical nonlinearities were canceled suggests a more complex story, which remains an area of open study.

4.2. Bifurcations and multistability

After the onset of the Duffing nonlinearity, a portion of the frequency response splits into two meta-stable branches and becomes double-valued. The frequencies at which the jumps between branches occur are called the bifurcation points and operating at these points leads to a number of interesting phenomena.

Early work studying resonator dynamics at these bifurcation points was done by Stambaugh *et al* using torsional polysilicon resonators [71]. In these experiments, the resonators were driven at the high amplitude branch near a bifurcation point and noise was injected into the driving signal. Since the bifurcation points are meta-stable, noise injected into the system makes it possible for the resonator to escape the high amplitude state to the low amplitude state, with the average transition rate being exponentially dependent on the amplitude of the noise. Switching between these meta-stable states has also been investigated by Almog *et al* as a method for signal amplification, called stochastic resonance [72]. In this work, a resonator was driven at a point of equal transition rates between the high and low amplitude branches, and the drive signal was amplitude modulated. When the optimal amount of noise is injected into the driving signal, the resonator undergoes stochastic resonance. During stochastic resonance the resonator jumps from one state to the other in synchronization with the modulation signal, which greatly amplifies the displacement. These amplitude jumps near the bifurcation points can be utilized for nonlinear resonant sensors that potentially have much higher sensitivity than their linear counterparts, as was demonstrated by Kumar *et al* [73]. Linear resonant chemical sensors typically measure the shift in resonance frequency due to the mass loading of adsorbed particles. In the case of the nonlinear sensor developed by Kumar *et al*, the device is driven at a constant frequency at the low amplitude state near the bifurcation point. The output amplitude is measured, and particle adsorption is signaled by the transition to the high amplitude state.

In electrostatic silicon resonators, a dominant electrostatic nonlinearity leads to spring softening, while a dominant mechanical nonlinearity leads to spring hardening. For intermediate levels of DC bias ‘mixed behavior’ is possible, where large drive power causes a hardening response, but even larger drives cause a softening response as the electrostatic nonlinearity begins to dominate [21, 56, 58, 74, 75], as shown in figure 4. Because there is simultaneously spring hardening and softening, there are now two hysteresis loops within the frequency response and four bifurcation points. First analytically predicted [58], this regime of electro-mechanical nonlinearity was experimentally demonstrated in 2009 by Kacem *et al* [56].

Sobreviela *et al* demonstrated that mixed behavior can be used to mitigate A-f noise by operating at a top bifurcation point [74], allowing for higher signal levels than previous work on canceling the third-order nonlinearity [12, 14, 76]. Subsequent work using a closed-loop oscillator found that operating at the top mechanical and bottom electrical bifurcation points yielded Allen deviation superior to the linear case, leveraging

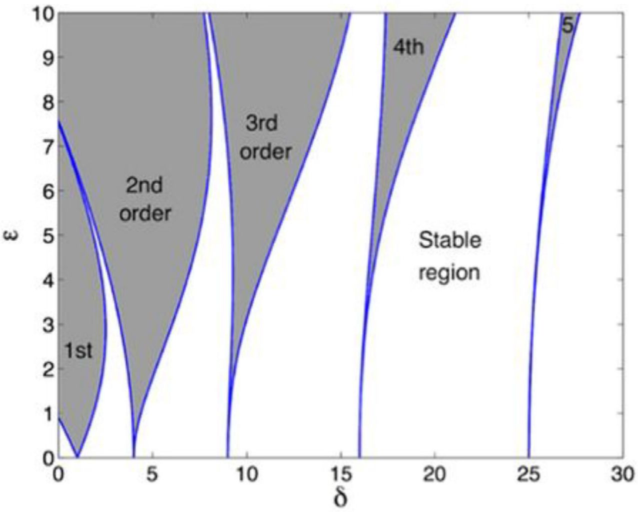

Figure 5. Plot showing stable regions for parametric system. δ represents the square of the natural frequency (ω_0^2), and ϵ is related to the amplitude of the parametric drive signal $\left(\frac{h}{2}\right)$. Reproduced from [79]. CC BY 4.0.

the simultaneous reduction of A-f noise and phase feedback noise (a result of the infinite value of $\frac{\partial \phi}{\partial f}$ at a bifurcation point) [75]. A closed-loop oscillator using a pair of coupled resonators has also been demonstrated, improving the SNR by 30 dB by operating at the top electrical nonlinearity [21]. Operating at this point improved stability for shorter integration times but was worse than the linear mode of operation for integration times longer than 0.1 s. This is because the higher order noise mixing processes are more significant in the nonlinear regime.

Nonlinear forces can lead to bi-stability in ways other than hysteresis in the frequency domain. For example, MEMS cantilevers with integrated permanent magnets have been shown to have two equilibrium bending states due to the nonlinear force from a neighboring permanent magnet [77]. When operated in this configuration, the cantilever loses the traditional Lorentzian frequency response and now has a low-pass filter type response. This device is designed to be a mechanical energy harvester, using this technique to improve the harvesting bandwidth.

4.3. Parametric processes

Parametric systems are generally defined by the presence of time varying parameters (e.g. mass, damping, stiffness). In the world of micro/nanosystems, parametric systems typically involve sinusoidal modulation of one of these parameters, usually the stiffness. A typical parametric system with a parametrically modulated stiffness can be described by [78]

$$\ddot{x} + b\dot{x} + \left(\omega_0^2 + h\cos\left(\omega_p t\right)\right)x + \beta x^3 + \eta x^2\dot{x} = 0, \quad (6)$$

where ω_0 is the natural frequency of the resonator, and ω_p is the parametric drive frequency, which is driven with amplitude h, and η is the coefficient of nonlinear damping. Solutions to this equation define regions of natural frequency and parametric drive amplitude where parametric excitation can be achieved (figure 5).

Use of parametric amplification in micro/nanosystems is frequently of interest for sensing applications because extreme sensitivity is possible between stable and unstable regions of parameter space. For sensing, a large jump in amplitude occurs in the transition from the unstable to the stable regions. This transition can be induced with a very small change in one of the parameters (e.g. mass). The review papers by Rhoads *et al* in 2008 and 2010 thoroughly cover parametrically-excited systems up to that point [3, 5], and this section will focus on work that occurred following that review.

In the area of parametric sensing, early work assessed the potential of parametric MEMS devices for mass sensing, focusing on how mechanical and electrical nonlinearities affect the stability regions of parametric oscillation [80]. Subsequent work from the same research group used parametric amplification to detect trace amounts of chemicals with a limit of detection of 13.3 parts per trillion [81]. The researchers combined knowledge of the system dynamics with a scheme that keeps a stable vibration amplitude by changing the frequency. The change in frequency is used as an indicator of adsorbed or desorbed mass. In the space of inertial sensors, Nitzan *et al* used geometric stiffness nonlinearity to parametrically amplify the Coriolis force via modulation of the sense axis, thereby creating a sensor whose parametric amplification was self-induced (i.e. no external parametric driving signal is required) [82].

Parametric control can also be used for device calibration. Lajevardi *et al*, used control of spring softening to calibrate drift in commercial accelerometers [83]. Calibration of accelerometers in the field is a vexing problem because it is extremely difficult to 'turn off' the input of acceleration for the purposes of calibration. To solve this problem, the authors electrostatically modulated the stiffness of the device, allowing them to distinguish between actual acceleration and parasitic capacitance.

The Roukes group demonstrated parametric amplification in nanomechanical devices in 2009 [84] Following that initial demonstration, the group began investigating methods to use parametric driving to reduce phase noise in oscillators with initial theoretical work [70]. Following that, Villanueva *et al* experimentally demonstrated an oscillator topology that uses nonresonant parametric feedback [85]. The oscillator induces motion in the device by modulating a physical parameter of the resonator at $2\omega_0$, as opposed to direct drive at ω_0. In this work, nonlinear stiffness is modulated by controlling the gain and phase from the feedback circuit. The purported advantages of this type of system over oscillators with direct drive are increased drive amplitude and frequency tunability. The authors observed improvements in frequency stability in phase noise for this oscillator as compared to an oscillator based on direct drive, and they noted that a combination of direct and parametric drive could potentially be used to further improve frequency stability.

Other groups have also pursued investigations in frequency stabilization and frequency tuning in parametric systems. In the field of frequency stabilization, Kacem *et al* used a parametric effect to mitigate the 5th order electrical softening nonlinearity [59]. The authors used direct drive of resonators, but included a superharmonic driving component at twice the resonant frequency to cancel the nonlinearity. For frequency tuning, Shmulevich *et al*, used a tapered comb drive architecture to create a parametric resonator whose frequency could be tuned by a DC voltage but would be unaffected by the amplitude of the structure's motion [86]. Additional work by the same group showed the device architecture was frequency tunable without any distortion of the shape of the instability windows due to this tapered structure, [87] and higher order resonances were subsequently investigated [88].

Work by Jia *et al* investigated the use of parametric resonance for frequency tunability and bandwidth, with the goal of increasing the bandwidth of vibration energy harvesters [79]. In this work, the authors use piezoelectrically actuated membranes with a proof mass in the middle of the membrane. The device is actuated to a sufficient amplitude, such that the nonlinearity of the membrane stiffness is significant; this modulation of the stiffness with membrane displacement drives the parametric resonance. Additional motivation for this work was scientific exploration of higher orders of parametric resonance (up to 28 were observed in this work for undamped membranes), as this work and several others have noted the ability of micro/nanoscale systems to access higher order modes of parametric resonance typically not seen in their macroscale counterparts [3, 89].

As opposed to other work that applied a stimulus to control where on the stability diagram the devices was operating, work by Karabalin *et al* manipulated the shape of the diagram itself around an existing stimulus [78]. This was achieved using two coupled nanomechanical resonators and modulating the stability diagram by applying a voltage that controlled the coupling strength between the two resonators. The authors note that this method for modulating the coupling could be used as an extremely sensitive charge detector or for other applications that involve sensing oscillating electric fields.

4.4. Frequency mixing and conversion

A fundamental property of linear systems is that the response frequency is identical to the driving force frequency. Any frequency mixing must then come from a nonlinearity in the system, which has both advantages and disadvantages. For example, frequency mixing can enable extremely low noise measurements of a sensor [28]. Frequency mixing can be also useful when it enables higher frequency signals in communications networks to be down-converted to a comfortable range of resonant frequencies for micromechanical devices [90]. On the other hand, in communication applications frequency conversion could have a detrimental effect if it allows out of band interferers to shift into the band of interest [15].

As mentioned in the previous paragraph, frequency mixing can be utilized to enable extremely low noise measurements of sensor displacement. One example that is widely used is electrostatic amplitude modulation (EAM). EAM leverages the fact that the electrostatic force varies quadratically with the applied voltage, $F \propto V^2$. By placing an AC carrier signal on top of the DC bias, the output signal can be modulated to be a different frequency than the parasitic feedthrough of the drive signal. Using this method is more complicated in devices with large displacements, since electrostatic nonlinearities arise in the parallel plate capacitances and require more complex analysis [28].

More recent work used this type of mixing to investigate the fundamental limits of frequency fluctuation [91]. By mixing down the output signal to a lower frequency, the authors were able to mitigate the effects of parasitic capacitance while still avoiding $1/f$ noise. Other physical mechanisms can also be used for mixing. Magnetoelastic transduction in micromechanical systems has been investigated for its potential to control magnetic fields at small scales. In one example, a cantilever heterostructure of magnetic and piezoelectric thin films was used to demonstrate magnetoelastic frequency doubling [44]. The magnetic film was poled in such a way that an applied RF magnetic field with frequency ω would create a mechanical vibration, and a corresponding output voltage from the piezoelectric, at 2ω. Potential applications of this approach include miniature RFID tags that are immune to reflected signals and noise isolation for measurement of magnetoelectric-based antennas.

Nonlinear mixing in a single structure can also lead to interesting output. For example, Ganesan *et al* demonstrated a phononic frequency comb via parametric coupling by driving the device off resonance with sufficient power [92]. Subsequent work by the same group investigated the space of multiple multimode regimes for these phononic frequency combs [93].

4.5. Nonlinearly coupled resonances

Linearly coupled MEMS resonators have been well established [94–96], with applications such as filters and sensors. The aforementioned frequency mixing from nonlinear processes gives rise to new possibilities for coupled resonators, allowing for resonances far apart in frequency to couple.

Mahboob *et al* investigated the dynamics of a micromechanical resonator coupled with a nanomechanical resonator embedded inside of it [97]. When driven into resonance, the nanoresonator modifies the local tension in the larger microresonator, lowering the spring constant and resonant frequency of the microresonator through the nonlinear mechanical coupling. This change in the microresonator frequency response is found to vary quadratically with the displacement of the nanoresonator, even when the nanoresonator is driven past the onset of the Duffing nonlinearity.

Frequency mixing from nonlinearity allows a resonator to couple and excite other resonance modes within itself, a process called 'internal resonance' [20, 98] (figure 6). For example, it has been demonstrated that a silicon resonator with a hardening cubic nonlinearity will increase in resonant frequency with drive power until its peak amplitude shifts to a frequency 1/3 that of a higher frequency mode [20]. At this point the spring hardening immediately stops, as any additional drive power serves to drive the higher frequency mode instead of the primary mode. As a consequence of the higher

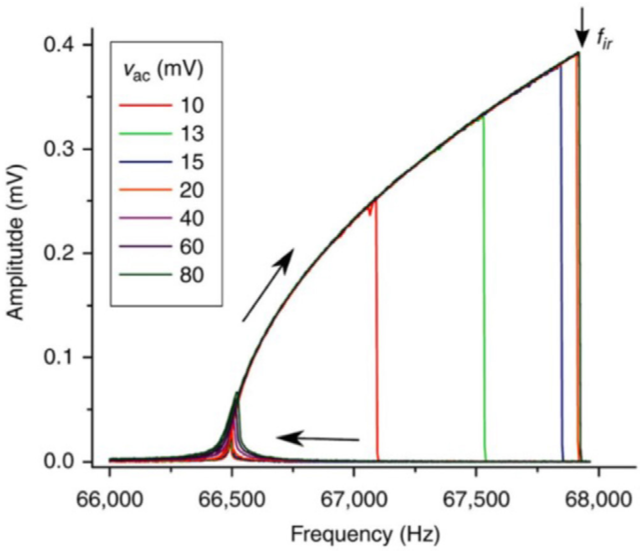

Figure 6. Internal resonance stabilizing spring hardening behavior. Reproduced from [20].

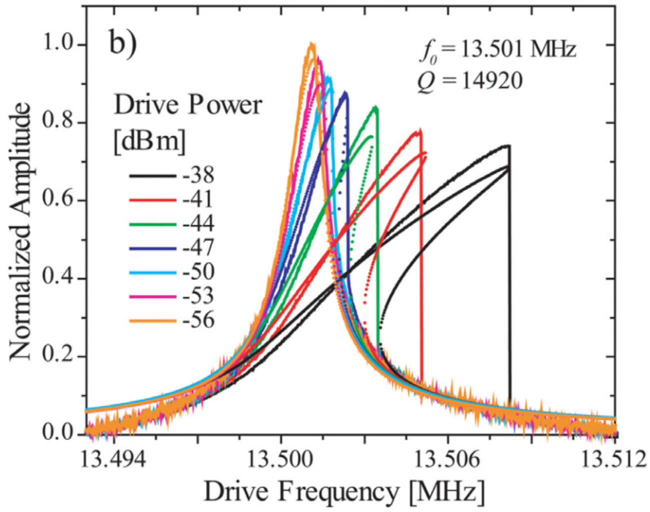

Figure 7. Normalized amplitude with different drive powers for a resonator with nonlinear dissipation. Reprinted from [103] with the permission of AIP Publishing.

frequency mode absorbing any amplitude fluctuations, the frequency stability of the primary mode is greatly increased. The authors observed a window of amplitude values where the minimum Allan deviation of the fundamental mode was improved by two orders of magnitude. A large enough drive power will overcome the internal resonance and cause the spring hardening to resume. The higher frequency mode can also act as a sort of rechargeable battery [98]. When driven to internal resonance, energy from the primary mode is transferred to the higher frequency mode. When the excitation is shut off, this energy is now transferred back to the primary mode, which allows the mode to keep a steady amplitude for a coherence time set by the original drive signal strength.

Sarrafan *et al* investigated using geometric nonlinearity to couple two normally decoupled modes in a tuning fork resonator [99]. Unlike the previously mentioned instances of internal resonance, this work utilizes subharmonic excitation. A translational ‘spring’ mode is driven to a large displacement, which causes appreciable rotation of the anchor. The normally rigid anchor now serves as a way to excite internal resonance, pumping energy to a rotational ‘pendulum’ mode that has a frequency half that of the spring mode.

To investigate nonlinear mode-coupling in NEMS devices, Matheny *et al* developed a characterization method with a high degree of linearity and sensitivity by using piezoelectric excitation and piezoresistive sensing [35]. This method was used to characterize the mode-coupling of the sub-harmonic and super-harmonic excitations, finding excellent agreement for the nonlinear coefficients measured and those predicted through Euler–Bernoulli beam theory. Modes do not have to be exact integer harmonics of each other to couple. Li *et al* demonstrated a method of coupling two modes together using an external pump signal and applied it to carbon nanotube resonator [100]. The frequency of the pump is set to be the frequency difference between the modes to be coupled and the coupling strength is determined by the pump amplitude. For a weak pump power of −25 dBm the frequency response remains unchanged. With a pump of >−20 dBm, the two modes begin to significantly couple, and normal mode splitting, characteristic of coupled oscillators, was observed.

4.6. Nonlinear dissipation processes

Nonlinear energy dissipation, distinguished by a drive amplitude dependence of the damping, is a relatively unexplored area in MEMS nonlinearity when compared to our previously discussed topics [101, 102]. A common way to model nonlinear dissipation is with the van der Pol–Duffing equation [56, 101–105]:

$$m\ddot{x} + b\dot{x} + kx + \beta x^3 + \eta x^2 \dot{x} = F_{drv}. \tag{7}$$

Here nonlinear damping is represented by the coefficient η. As with all nonlinear processes, characterization of nonlinear dissipation processes is a challenging issue. One method is to measure the device at several drive powers, normalizing each output by the drive power [103]. Nonlinear coefficients can then be extracted by measuring how much the normalized amplitude decreases as drive power is increased (figure 7). Polunin *et al* developed a characterization method using only a single ring-down measurement, bypassing the need for several measurements at different drive powers in typical nonlinear characterization [104, 105]. Using this approach, they are able to determine the nonlinear damping coefficient, as well as the cubic and 5th order nonlinear spring constant terms.

Despite the wide use of the van der Pol–Duffing model, the actual mechanisms for nonlinear damping are varied and often poorly understood. In a study by Zaitsev *et al* using PdAu doubly-clamped beams, no clear mechanism for the nonlinear damping is found but it is proposed that geometric nonlinearity is a contributing factor, with linear damping from the material stiffness becoming nonlinear through the resonator’s geometric nonlinearity [106]. Squeeze-film damping is a nonlinear process as well, presenting an issue for energy harvesters that have large displacements [107]. Work on an

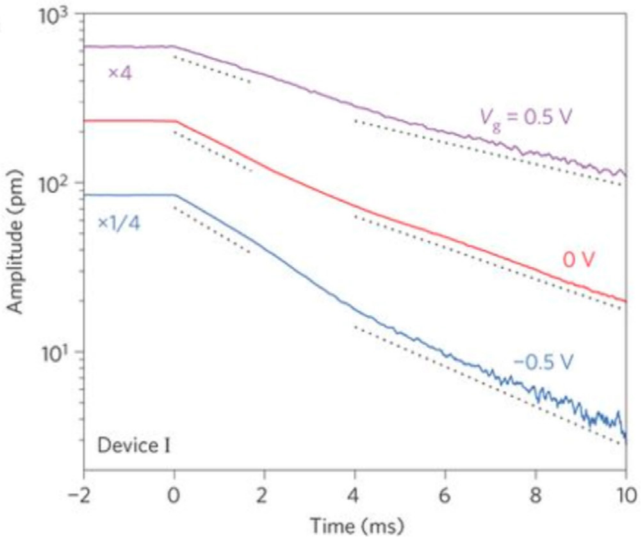

Figure 8. Decay of amplitude during ring down experiments by Güttinger *et al* [110]. Notice how the decay is steeper earlier in the measurement. [110] Copyright © 2017, Springer Nature. With permission of Springer.

atomic force microscope (AFM) cantilever with an embedded nanotube found that nonlinear damping in the nanotube to be significant for the system as a whole, with the magnitude of the damping being consistent with viscoelastic domain wall loss in a multi-domain nanotube [31]. In a study on the effect of metallic coatings on MEMS cantilevers, it was found that the coatings introduce nonlinear damping into the device, though the actual mechanism is not known [108]. Nonlinear damping has also been shown to be temperature dependent, with work by Imboden *et al* on diamond resonators showing a transition from linear to nonlinear damping below 77 K [109].

Nonlinearity can affect damping in ways outside of drive amplitude dependence. For example, it has been demonstrated that a resonance coupled to a higher frequency mode through internal resonance can have an increased damping rate since both modes dissipate the energy (figure 8) [110]. During the ring-down experiments, it was that found past a critical amplitude the decay rate becomes that of the primary mode as the nonlinear coupling between modes is now too weak. Miao *et al* studied the temperature dependence of the quality factor in graphene membrane resonators and found that the predominant source of damping in from the linewidth broadening caused by nonlinear coupling to thermally excited modes [111]. It has also been shown that nonlinear damping can also be accidentally induced by phase errors in feedback loops when working with a nonlinear resonator [112].

5. Conclusion

In the past decade, nonlinearities in MEMS devices have seen increased interest across several fronts. Early research in MEMS resonators focused on canceling nonlinearities to improve the drive current, but were later leveraged to optimize the phase noise and frequency stability in oscillators. Cancellation of nonlinearities was later applied to sensors, such as resonant gyroscopes, which demonstrated improved performance in some cases. Bifurcations caused by nonlinearities have been embraced, finding applications in the further enhancement of oscillator stability. Parametric processes in MEMS devices moved beyond initial demonstrations of parametric resonance to applications utilizing parametric processes to enhance performance, such as improved sensitivity in chemical sensors and increased bandwidth in energy harvesters. In the area of frequency mixing/conversion, use of inherent nonlinearity of electrostatic actuation has sustained its popularity for high resolution capacitive detection. Recently, the use of new materials and methods have emerged, such as frequency doubling via magnetoelastic materials and investigations of phononic frequency combs. Frequency mixing has been exploited to enable nonlinear coupling of different resonant modes, leading to new observations of phenomena, such as internal resonance. Research on nonlinear dissipation process has only begun to scratch the surface of otherwise very complex systems, with much more room to explore in the future. Arguably the most striking trend of the past decade is the evolving viewpoint on nonlinearities, from a nuisance that should be minimized to a feature that can be exploited for improved performance.

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